reefs, the fish of known species becoming larger as we go outward. Some of the most abundant mid-water types, such as *Chromis cyaneus* and *Sardinella aurita* are rare or unknown in-shore.

Beyond this and down to 200 feet, the bottom is less rugged, and the seafans and plumes smaller. Then comes an abrupt range; in one spot, over which the Bathysphere had to be drawn, 50 feet in height. Beyond this comes a long, sloping beach, of water-worn pebbles and stones, wholly without living growths or fish, and ending in wide stretches of white, rippled sand. It is probable that this is the beach of the last low ocean level of the glacial period.

The slow seaward drift of the tug, and the average level of the Bathysphere of 10 feet above the bottom, enable us to see all fish, down to two inches, with perfect clearness.

In mid-summer, with a day of brilliant sunshine and with clear water, it will be quite possible to follow the volcanic slopes down for many hundred feet farther without losing sight of the bottom.

[1] *N. Y. Zoöl. Soc. Bull.*, **33**, No. 6, 227.
[2] *Jour. Opt. Soc. Amer.*, **22**, No. 7, 408–417.
[3] The Arcturus Adventure, pp. 214–218.

A STEP-POLYGON OF A DENUMERABLE INFINITY OF SIDES WHICH BOUNDS NO FINITE AREA

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The problem of Plateau was completely solved by J. Douglas,[1] who proved the existence of a minimal surface bounded by any given Jordan curve.

The solution divided itself into two parts, the first referring to a contour which spans at least one surface of finite area, after which by a limit process the existence proof was extended to an arbitrary Jordan curve Γ which spans no finite area: every surface bounded by Γ has infinite area.

The question was thus raised as to the effective existence of the second type of contour. An example (P), based on Peano curves in the xy-plane, was constructed by the joint authors of this note,[2] and the basis of proof briefly indicated as consisting in infinite area of the horizontal projection of (P). A few further remarks concerning this example will be found at the end of this note.

Our main purpose in this communication is to give a simpler example in the form of a step-polygon of a denumerable infinity of sides, together with all the details of proof. This step-polygon may be considered as more simple than, for instance, many rectifiable curves—showing that one must be careful not to overestimate the generality of any treatment of the Plateau problem that restricts itself to the case of a finite-area-spanning contour.

1. *Description.*—To arrive at the example, we begin by constructing a "skyscraper" of an infinite number of floors, and with infinite floor space, but finite height. The ground plan (horizontal projection) of the skyscraper is shown in figure 1. The sides of the squares, which crowd into the left rear corner, are

$$\frac{1}{\sqrt{n}} \quad (n = 1, 2, 3, \ldots \text{ ad infin}),$$

and each square represents a floor. The total floor space is the sum of the divergent harmonic series:

$$\sum_{n=1}^{\infty} \frac{1}{n} = +\infty.$$

The skyscraper is formed by raising the nth square a vertical distance

$$\frac{1}{2^{n-1}},$$

so that the total height is unity.

We next take the nth floor and divide it into

$$2^n + 1$$

equal rectangular strips. Then we construct a broken line as indicated in figure 2, running back and forth along the sides of these rectangles. Since $2^n + 1$ is odd, it is evident from the mode of construction that the broken line will end at the left forward corner B_n of the square if it starts at the left rear corner A_n.

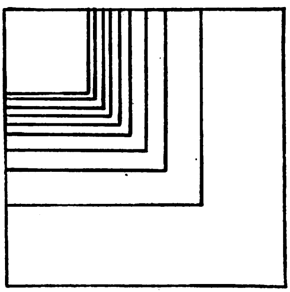

FIGURE 1

We thus have a rectangular broken line in each floor. These we connect by joining A_{2m} to A_{2m-1} with a vertical line segment, and B_{2m} to B_{2m+1} with a vertical segment $B_{2m}C_{2m+1}$ to the level of the $(2m + 1)$th floor followed by the horizontal segment $C_{2m+1}B_{2m+1}$.

We now have a Jordan arc in the form of a rectangular broken line consisting of a denumerable infinity of segments, going from B_1 (0,1,0) to A_∞ (0,0,1). This may be completed to an infinitely many sided closed Jordan step-polygon by the line segments B_1C, CA_∞, where C is the point (0,1,1).

Our example is now finished.

2. *Proof.—Cylindrical Pipes.*—About each of the transverse, or long, segments of the broken line which traverses the nth square—except the last segment, ending at B_n—let a cylinder of revolution be constructed as follows. The axis of the cylinder shall be one-half the length of the segment, symmetrically placed with respect to the ends (from the one-quarter division point to the three-quarters point); the radius R_n shall be one-half the width of a rectangular strip, that is:

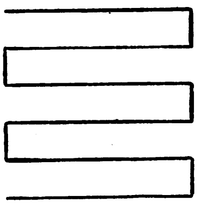

FIGURE 2

$$R_n = \frac{1}{2(2^n + 1)\sqrt{n}}.$$

Since the distance between the nth and $(n + 1)$th floors is $\frac{1}{2^n}$, each cylinder reaches less than half-way to the floor next above, and *a fortiori* less than half-way to the floor next below; therefore the cylinders *do not overlap,* though (what is of no importance) those on the same floor are externally tangent.

We shall need the fact that the total area of the generating rectangles of the cylinders on the nth floor is one-fourth the area of that floor, $= \frac{1}{4n}$.

Next we need the following lemma.

LEMMA. *Consider a right circular cylinder with axis AB. If any simply connected surface is spanned by AB together with a curve joining A and B lying entirely outside the cylinder, the area of the portion of the surface interior to the cylinder is at least equal to that of the generating rectangle of the cylinder.*

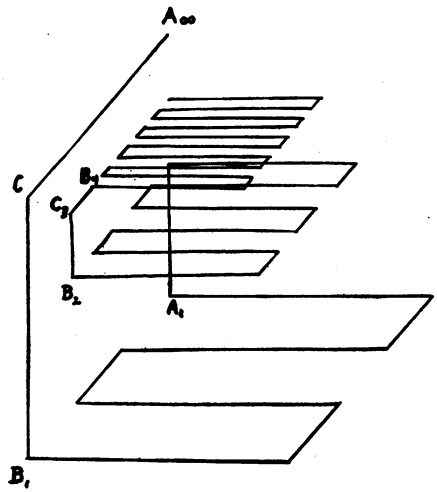

FIGURE 3

The proof results immediately by circular projection of the portion of surface interior to the cylinder on a meridian plane, an operation which obviously diminishes area (or, at most, leaves it the same), multiplying each element of area by the cosine of the angle between the tangent plane to the element and the meridian plane through the point where the element is located. By the same proof, the theorem extends to any solid of revolution, the minimum intercepted area being half a meridian section.

Infinite Area.—Consider now the cylindrical pipes on the nth floor.

The sum of the areas of their generating rectangles is, as has been remarked, $\frac{1}{4n}$; consequently the portion of any simply connected surface Σ bounded by the step-polygon intercepted by these cylinders is at least $\frac{1}{4n}$ in area. Since the cylinders on different floors do not overlap, the area of the portion of Σ intercepted by all the cylinders is at least $\frac{1}{4}$ the sum of the divergent harmonic series; hence

$$\textit{Area of } \Sigma = +\infty,$$

which was to be proved.

The infinite part of the area of the minimal surface M determined by the step-polygon is all bunched up in the vicinity of the singular point A_∞, summit of the "skyscraper." If, in traversing the boundary of M, we make a detour around A_∞, passing between two boundary points on opposite sides of A_∞ by any arc lying, except for its end-points, in the interior of M, then the area of the intercepted part of M will be finite.

3. *Remarks on the Example* (P).—If the reader is interested in the details of proof for the originally given contour (P), he should find little difficulty in supplying them on the model of the foregoing discussion. In fact, if the Peano curve (P') which is the xy-projection of (P) is replaced by a sufficiently advanced polygonal approximation,[3] the following hint should suffice to prove that even the simplified version of (P) so resulting (skew polygon of a denumerable infinity of sides) can span no finite area.

Hint.—A square column, tall, with very small base, is traversed by a line segment σ from a point well up on a lateral edge to a point at about the same level on the diagonally opposite edge. Then the area intercepted by the column on any surface bounded by a contour consisting of σ together with an arc lying entirely outside the column is at least equal to the area of one of the 45° right triangles into which the base is divided by a diagonal.

[1] "Solution of the Problem of Plateau," *Trans. Amer. Math. Soc.*, **33**, 263–321 (1931).

[2] Loc. cit., p. 320.

[3] F. Klein, *Elementarmathematik vom höheren Standpunkte aus*, 3d ed., Berlin, 1928, p. 119.